

# ON PRINCIPAL MINORS OF BEZOUT MATRIX

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## Abstract

Let  $x_1, \dots, x_n$  be real numbers,  $P(x) = p_n(x-x_1)\cdots(x-x_n)$ , and  $Q(x)$  be a polynomial of degree less than or equal to  $n$ . Denote by  $\Delta(Q)$  the matrix of generalized divided differences of  $Q(x)$  with nodes  $x_1, \dots, x_n$  and by  $B(P, Q)$  the Bezout matrix (Bezoutiant) of  $P$  and  $Q$ . A relationship between the corresponding principal minors, counted from the right-hand lower corner, of the matrices  $B(P, Q)$  and  $\Delta(Q)$  is established. It implies that if the principal minors of the matrix of divided differences of a function  $g(x)$  are positive or have alternating signs then the roots of the Newton's interpolation polynomial of  $g$  are real and separated by the nodes of interpolation.

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## 1 Introduction.

In this paper a relationship between two well known matrices is established. The first one is a Bezout matrix  $B$  playing an important role in the theory of separation of polynomial roots. The second one is Newton's matrix of divided differences  $\Delta$  or, in the case of

multiple nodes, Hermite's matrix of generalized divided differences, playing an important role in numerical analysis and approximation theory. In this paper we show that the corresponding principal minors of  $B$  and  $\Delta$  counted from the right-hand lower corner are related by a simple formula (are equal when  $p_n = 1$ ). An alternative proof of this result can be obtained from the results of [10]. As a simple application of the relationship between  $B$  and  $\Delta$ , a theorem about locations of the roots of interpolation polynomials in terms of the principal minors of  $\Delta$  is established. Many applications of Bezout matrix can be found in [1]-[11]. The results of this paper were announced without proofs in [12].

## 2 Main Results.

With the polynomials  $P(x) = \sum_{j=0}^n p_j x^j$  and  $Q(x) = \sum_{j=0}^n q_j x^j$  let us associate the bilinear form

$$\sum_{i,j=1}^n b_{ij} x^{i-1} y^{j-1} = \frac{P(x)Q(y) - P(y)Q(x)}{x - y}, \quad (1)$$

which Sylvester [1] named "Bezoutiant". If the degree of  $Q$  is less than the degree of  $P$ , that is  $Q(x) = \sum_{j=0}^m q_j x^j$ ,  $m < n$ , then one adds zero coefficients  $q_{m+1}, \dots, q_n$  to  $Q$ . In what follows we assume that  $m \leq n$  and denote  $B(P, Q) = ||b_{ij}||_{i,j=1,\dots,n}$ .

It has been shown (see [7]) that

$$B(P, Q) = \begin{pmatrix} p_1 & p_2 & \cdot & \cdot & \cdot & p_n \\ p_2 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ p_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} q_0 & \cdot & \cdot & \cdot & q_{n-2} & q_{n-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & q_{n-2} \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & \cdot & & q_0 & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & q_0 \end{pmatrix} \\ - \begin{pmatrix} q_1 & q_2 & \cdot & \cdot & \cdot & q_n \\ q_2 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ q_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} p_0 & \cdot & \cdot & \cdot & p_{n-2} & p_{n-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & p_{n-2} \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & \cdot & p_0 & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & p_0 \end{pmatrix}. \quad (2)$$

The main properties of the Bezoutiant are (see [5, 7, 8, 9]):

- The defect of the Bezoutiant equals the degree of the greatest common divisor of the polynomials  $P$  and  $Q$ .
- The rank of the Bezoutiant matrix equals the degree of the last principal minor of the matrix  $B = ||b_{i,j}||_{i,j=1,\dots,n}$  which does not vanish if, in constructing the consecutive major minors, one starts from the lower right-hand corner.
- If the Bezoutiant matrix is positive definite then both polynomials  $P(x)$  and  $Q(x)$  have real, distinct roots. Moreover, the roots of  $P(x)$  and  $Q(x)$  interlace.
- If all consecutive principal minors starting from the lower right-hand corner are positive or have alternating signs, then the roots of  $P(x)$  and  $Q(x)$  are real, distinct, and interlace.

Since principal minors of Bezoutiants play so important a role, it seems interesting to find explicit formulas for them. If the roots  $x_1, x_2, \dots, x_n$  of  $P(x)$  are simple, such formulas were established in [13].

**Theorem 1** *Let  $|b_{ij}|_{i,j=k+1}^n$  be the principal minors counted from the lower right corner of the Bezoutiant  $B(P, Q)$  of polynomials  $P(x) = p_n(x - x_1) \dots (x - x_n)$  and  $Q(x)$ . Then,*

$$\begin{aligned}
 |b_{i,j}|_{i,j=k+1}^n &= p_n^{2(n-k)} \sum_{\substack{(i_1, \dots, i_{n-k}) \subset (1, \dots, n) \\ i_1 < i_2 < \dots < i_{n-k}}} \frac{Q(x_{i_1}) \dots Q(x_{i_{n-k}})}{P'(x_{i_1}) \dots P'(x_{i_{n-k}})} \\
 &\times \prod_{\substack{(j_1, j_2) \subset (i_1, \dots, i_{n-k}) \\ j_1 < j_2}} (x_{j_1} - x_{j_2})^2. \tag{3}
 \end{aligned}$$

**Remark 1** *If  $k = n - 1$  then formula (3) becomes*

$$b_{n,n} = p_n^2 \sum_{i=1}^n \frac{Q(x_i)}{P'(x_i)}. \tag{4}$$

**Remark 2** Since  $|b_{ij}|_{i,j=k+1}^m$  are continuous functions of  $x_1, \dots, x_m$  in case of multiple roots one has to find the corresponding limit which is technically difficult and leads to complicated expressions.

In order to consider the case of  $x_1, \dots, x_n$  which are not necessarily different, let us introduce the following generalized divided differences.

**Definition 1** (see [14]):

$$g[x_i] := g(x_i), \quad i = 1, \dots, n,$$

$$g[x_{i_1}, \dots, x_{i_k}] := \begin{cases} \frac{g[x_{i_2}, \dots, x_{i_k}] - g[x_{i_1}, \dots, x_{i_{k-1}}]}{x_{i_k} - x_{i_1}}, & \text{if } x_{i_1} \neq x_{i_k} \\ \frac{d}{dx}g[x, x_{i_2}, \dots, x_{i_{k-1}}]_{x=x_{i_1}}, & \text{if } x_{i_1} = x_{i_k}. \end{cases} \quad (5)$$

**Remark 3** This definition of generalized divided differences is equivalent to the definition given in [14] if  $x_1 \leq x_2 \leq \dots \leq x_n$ .

Consider the following triangular matrix of the generalized divided differences:  $\Delta(g) = ||\Delta_{ij}||_{i,j=1,\dots,n}$ , where

$$\Delta_{ij} = \begin{cases} 0, & \text{if } i + j < n + 1 \\ g[x_{n-i+1}, \dots, x_j], & \text{if } i + j \geq n + 1, \end{cases} \quad (6)$$

that is

$$\Delta(g) = \begin{pmatrix} & & & 0 & \Delta_n \\ & 0 & & \Delta_{n-1} & \Delta_{n-1,n} \\ & & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \Delta_1 & \Delta_{1,2} & \cdot & \cdot & \Delta_{1,n-1} & \Delta_{1,n} \end{pmatrix}. \quad (7)$$

**Remark 4** As it is well known, Newton-Hermite's interpolation polynomial for  $n$  nodes  $\{x_1, \dots, x_n\}$  is  $\Delta_1 + \Delta_{1,2}(x - x_1) + \dots + \Delta_{1,n}(x - x_1) \dots (x - x_{n-1})$ .

Denote by  $|\Delta_{i,j}|_{i,j=k+1}^n$  the principal minors of the matrix  $\Delta$  counted from the lower right corner. The following theorem establishes a relationship between principal minors of the Bezoutiant and Newton's matrix.

**Theorem 2** Let  $|b_{i,j}|_{i,j=k+1}^n$  and  $|\Delta_{i,j}|_{i,j=k+1}^n$  be the principal minors of the matrices  $B(P, Q)$  and  $\Delta(Q)$  counted from the lower right corner. Then

$$|b_{i,j}|_{i,j=k+1}^n = p_n^{n-k} |\Delta_{i,j}|_{i,j=k+1}^n, \quad k = 0, 1, \dots, n-1. \quad (8)$$

The relationship between  $B(P, Q)$  and  $\Delta(Q)$  established in this theorem is surprising taking into account that these matrices are of very different type, the Bezoutiant is a symmetric matrix and Newton's matrix is a triangular matrix. Two simple examples below show these matrices for some polynomials of degree three.

**Example 1** Let us consider polynomials  $P(x) = x^3 - 4x^2 - x + 4 = (x+1)(x-1)(x-4)$  and  $Q(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$ . Then

$$B(P, Q) = \begin{pmatrix} -38 & 48 & -10 \\ 48 & -60 & 12 \\ -10 & 12 & -2 \end{pmatrix}, \quad \Delta(Q) = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 2 \\ -24 & 12 & -2 \end{pmatrix}.$$

Since  $p_3 = 1$  the corresponding principal minors of these two matrices counted from the lower right-hand corner are equal, they are  $-2, -24, 0$ .

**Example 2** Consider polynomials  $P(x) = x^3 - 12x^2 + 44x - 48 = (x-2)(x-4)(x-6)$  and  $Q(x) = x^3 - 9x^2 + 23x - 15 = (x-1)(x-3)(x-5)$ . Then

$$B(P, Q) = \begin{pmatrix} 444 & -252 & 33 \\ -252 & 153 & -21 \\ 33 & -21 & 3 \end{pmatrix}, \quad \Delta(Q) = \begin{pmatrix} 0 & 0 & 15 \\ 0 & -3 & 9 \\ 3 & -3 & 3 \end{pmatrix}.$$

Principal minors counted from the lower right-hand corner are 3, 18, 135.

Theorem 2 and the properties of the Bezoutiant described above imply the following theorem.

**Theorem 3** If all consecutive principal minors of the matrix of divided differences (see (7)) of some function  $g(x)$  starting from the lower right-hand corner are positive or have alternating signs, then the roots of Newton's interpolation polynomial are real, distinct, and interlace with the nodes of interpolation.

### 3 Proofs.

As it is shown in [7],

$$(-1)^{\frac{(n-k)(n-k-1)}{2}} |b_{i,j}|_{i,j=k+1}^n$$

$$= \begin{vmatrix} p_n & \cdot & \cdot & \cdot & \cdot & p_{k+1} & & & & & p_k & \cdot & \cdot & \cdot & \cdot & p_{2k-n+1} \\ 0 & p_n & \cdot & \cdot & \cdot & p_{k+2} & & & & & p_{k+1} & \cdot & \cdot & \cdot & \cdot & p_{2k-n+2} \\ \cdot & \cdot & \cdot & & & \cdot & & & & & \cdot & & & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot & & & & & \cdot & & & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & & & \cdot & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & p_n & & & & p_{n-1} & \cdot & \cdot & \cdot & \cdot & p_k \\ - & - & - & - & - & - & - & & & & - & - & - & - & - & - \\ q_n & \cdot & \cdot & \cdot & \cdot & q_{k+1} & & & & & q_k & \cdot & \cdot & \cdot & \cdot & q_{2k-n+1} \\ 0 & q_n & \cdot & \cdot & \cdot & q_{k+2} & & & & & q_{k+1} & \cdot & \cdot & \cdot & \cdot & q_{2k-n+2} \\ \cdot & \cdot & \cdot & & & \cdot & & & & & \cdot & & & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot & & & & & \cdot & & & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & & & \cdot & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & q_n & & & & q_{n-1} & \cdot & \cdot & \cdot & \cdot & q_k \end{vmatrix}.$$

Since  $Q$  is a polynomial of degree  $m$ ,  $|b_{i,j}|_{i,j=k+1}^n = 0$  for  $k = m + 1, \dots, n - 1$ . Thus, let us assume that  $k \leq m$  and  $k \leq n - 1$ . Then,

$$(-1)^{\frac{(n-k)(n-k-1)}{2}} |b_{i,j}|_{i,j=k+1}^n = p_n^{n-m} d, \quad (9)$$

where

$$d = \begin{vmatrix} p_n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_{2k-m+1} \\ 0 & p_n & \cdot & \cdot & \cdot & \cdot & \cdot & p_{2k-m+2} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & \cdot & \cdot & \cdot & 0 & p_n & \dots & p_k \\ q_m & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_{2k-n+1} \\ 0 & q_m & \cdot & \cdot & \cdot & \cdot & \cdot & q_{2k-n+2} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & \cdot & \cdot & \cdot & 0 & q_m & \dots & q_k \end{vmatrix}.$$

This determinant can be represented as,

$$d = \left| \begin{array}{cccc|cccc|cccc|cccc} p_n & \cdot & \cdot & \cdot & p_{n+k-m+1} & \cdot & \cdot & \cdot & p_0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot & & & & \cdot & & & & \cdot \\ \cdot & & \cdot & & \cdot & & & & & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & & & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & p_n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_0 \\ \hline q_m & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & & & & \cdot & & & & \cdot \\ \cdot & & \cdot & & & & & & & & \cdot & & \cdot \\ \cdot & & & \cdot & & & & & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & q_m & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_0 \\ \hline 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & \end{array} \right|, \quad (10)$$

where  $I_k$  is  $k \times k$  unit matrix (obviously, there are no rows below the second dashed line if  $k = 0$ ).

First let us assume that the roots of the polynomials  $P$  and  $Q$  are simple and distinct.

Denote by  $V_j(x_1, \dots, x_n, y_1, \dots, y_{n-k})$  the following matrix:

$$\begin{pmatrix} x_1^j & \dots & x_n^j & y_1^j & \dots & y_{m-k}^j \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix}$$

Then  $V_{n+m-k-1}(x_1, \dots, x_n, y_1, \dots, y_{n-k})$  is the Vandermont matrix and

$$\begin{aligned} & \det(V_{n+m-k-1}(x_1, \dots, x_n, y_1, \dots, y_{m-k})) \\ &= \prod_{1 \leq i_1 < i_2 \leq n} (x_{i_1} - x_{i_2}) \prod_{\substack{1 \leq i_1 \leq n \\ 1 \leq i_2 \leq m-k}} (x_{i_1} - y_{i_2}) \prod_{1 \leq i_1 < i_2 \leq m-k} (y_{i_1} - y_{i_2}) \\ &= \frac{(-1)^{n(m-k)}}{p_n^{m-k}} f(y_1) \dots f(y_{m-k}) \prod_{1 \leq i_1 < i_2 \leq n} (x_{i_1} - x_{i_2}) \prod_{1 \leq j_1 < j_2 \leq m-k} (y_{j_1} - y_{j_2}). \end{aligned} \quad (11)$$

Multiplying determinants(10) and (11) one gets:

$$d \cdot V \text{ and } = \begin{vmatrix} 0 & M_1 \\ M_2 & 0 \\ V_{k-1}(x_1, \dots, x_n) & V_{k-1}(y_1, \dots, y_m), \end{vmatrix}$$

where

$$M_1 = \begin{pmatrix} y_1^{m-k-1}P(y_1) & \dots & y_{m-k}^{m-k-1}P(y_{m-k}) \\ \vdots & & \vdots \\ P(y_1) & \dots & P(y_{m-k}) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} x_1^{n-k-1}Q(x_1) & \dots & x_n^{n-k-1}Q(x_n) \\ \vdots & & \vdots \\ Q(x_1) & \dots & Q(x_n) \end{pmatrix}.$$

Therefore,

$$d \cdot V \text{ and } = (-1)^{n(m-k)} P(y_1) \dots P(y_{m-k}) D \prod_{1 \leq j_1 < j_2 \leq m-k} (y_{j_1} - y_{j_2}), \quad (12)$$

where

$$D = \begin{vmatrix} x_1^{n-k-1}Q(x_1) & \dots & x_n^{n-k-1}Q(x_n) \\ \vdots & & \vdots \\ Q(x_1) & \dots & Q(x_n) \\ x_1^{k-1} & \dots & x_n^{k-1} \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{vmatrix}, \text{ if } k \geq 1, \quad (13)$$

and

$$D = \begin{vmatrix} x_1^{n-1}Q(x_1) & \dots & x_n^{n-1}Q(x_n) \\ \vdots & & \vdots \\ Q(x_1) & \dots & Q(x_n) \end{vmatrix}, \text{ if } k = 0. \quad (14)$$

From (9), (11), and (12) one gets

$$|b_{i,j}|_{i,j=k+1}^n = \frac{(-1)^{(n-k)(n-k-1)/2} p_n^{n-k}}{\prod_{1 \leq i_1 < i_2 \leq n} (x_{i_1} - x_{i_2})} D. \quad (15)$$

Since the case  $k = 0$  is trivial, let us assume that  $k \geq 1$ . Subtracting from all rows of the matrix  $D$ , except of the  $(n-k)$ th row



and of the last row, the next row, multiplied by  $x_1$ , and from the  $(n - k)$ th row the last row multiplied by  $Q(x_1)$  one gets

$$D = \begin{vmatrix} 0 & x_2^{n-k-2}(x_2 - x_1)Q(x_2) & \dots & x_n^{n-k-2}(x_n - x_1)Q(x_n) \\ \vdots & & & \vdots \\ 0 & (x_2 - x_1)Q(x_2) & \dots & (x_n - x_1)Q(x_n) \\ 0 & Q(x_2) - Q(x_1) & \dots & Q(x_n) - Q(x_1) \\ 0 & x_2^{k-2}(x_2 - x_1) & \dots & x_n^{k-2}(x_n - x_1) \\ \vdots & \vdots & \dots & \vdots \\ 0 & (x_2 - x_1) & \dots & (x_n - x_1) \\ 1 & 1 & \dots & 1 \end{vmatrix}. \quad (16)$$

After pulling out the common multipliers  $x_j - x_1$ ,  $j = 2, \dots, n$  from columns, one obtains:

$$D = (-1)^{n+1} \prod_{2 \leq j \leq n} (x_j - x_1) \begin{vmatrix} x_2^{n-k-2}Q(x_2) & \dots & x_n^{n-k-2}Q(x_n) \\ \vdots & & \vdots \\ Q(x_2) & \dots & Q(x_n) \\ Q[x_1, x_2] & \dots & Q[x_1, x_n] \\ x_2^{k-2} & \dots & x_n^{k-2} \\ \vdots & \dots & \vdots \\ 1 & \dots & 1 \end{vmatrix}. \quad (17)$$

Now let us consider two cases:  $n - k \geq k - 1$  and  $n - k < k - 1$ . Denote  $Q[i] := Q(x_i)$ ,  $Q[i_1, i_2, \dots, i_k] := Q[x_{i_1}, x_{i_2}, \dots, x_{i_k}]$ .

**The first case.** If  $n - k \geq k - 1$ , then, continuing this process, after  $k$  steps one gets

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \prod_{\substack{1 \leq j_1 \leq k \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}) \times \begin{vmatrix} x_{k+1}^{n-2k-1}Q[k+1] & \dots & x_n^{n-2k-1}Q[n] \\ \vdots & & \vdots \\ Q[k+1] & \dots & Q[n] \\ Q[k, k+1] & \dots & Q[k, n] \\ Q[k-1, k, k+1] & \dots & Q[k-1, k, n] \\ \vdots & \dots & \vdots \\ Q[1, \dots, k, k+1] & \dots & Q[1, \dots, k, n] \end{vmatrix}. \quad (18)$$

Subtracting from the first  $n - 2k - 1$  rows the next row multiplied by  $x_{k+1}$  one obtains:

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \det(C) \prod_{\substack{1 \leq j_1 \leq k+1 \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}), \quad (19)$$

where the columns of the matrix  $C$  are:

$$C_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Q[k+1] \\ Q[k, k+1] \\ \vdots \\ Q[1, \dots, k, k+1] \end{pmatrix}, C_i = \begin{pmatrix} x_{k+i}^{n-2k-2} (x_{k+i} - x_{k+1}) Q[k+i] \\ \vdots \\ (x_{k+i} - x_{k+1}) Q[k+i] \\ Q[k+i] \\ Q[k, k+i] \\ Q[k-1, k, k+i] \\ \vdots \\ Q[1, \dots, k, k+2] \end{pmatrix},$$

$i = 2, \dots, n - k$ . (The notation  $C$  will be used below to denote different matrices.)

After subtracting the first column from the other columns and pulling out the common factors  $(x_{k+1} - x_k), \dots, (x_n - x_k)$ , one gets

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \prod_{\substack{1 \leq j_1 \leq k+1 \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}) \times$$

$$\begin{vmatrix} 0 & x_{k+2}^{n-2k-2} Q[k+2] & \dots & x_n^{n-2k-2} Q[n] \\ \vdots & \vdots & & \vdots \\ 0 & Q[k+2] & \dots & Q[n] \\ Q[k+1] & Q[k+1, k+2] & \dots & Q[k+1, n] \\ Q[k, k+1] & Q[k, k+2] & \dots & Q[k, n] \\ Q[k-1, k, k+1] & Q[k-1, k, k+2] & \dots & Q[k-1, k, n] \\ \vdots & \vdots & \dots & \vdots \\ Q[1, \dots, k, k+1] & Q[1, \dots, k, k+2] & \dots & Q[1, \dots, k, n] \end{vmatrix}. \quad (20)$$

Again, let us subtract from the first  $n - 2k - 2$  rows the next row multiplied by  $x_{k+2}$ , then, subtract the second column from the other columns, and pull out from the columns common factors  $(x_{k+2} -$

$x_{k+1}), \dots, (x_n - x_{k+1})$ . Continuing this process, after  $n - 2k - 1$  steps one gets:

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \det(C) \prod_{\substack{1 \leq j_1 \leq n-k-1 \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}),$$

where the columns of  $C$  are

$$C_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Q[k+i] \\ Q[k+i-1, k+i] \\ \vdots \\ Q[1, \dots, k+i] \end{pmatrix}, \quad i = 1, \dots, n-2k-1,$$

$$C_i = \begin{pmatrix} Q[k+i] \\ Q[n-k-1, k+i] \\ \vdots \\ Q[1, \dots, n-k-1, k+i] \end{pmatrix}, \quad i = n-2k, \dots, n-k.$$

Let us subtract from the last  $k$  columns the previous column and factor out  $\prod_{j=n-k+1}^n (x_j - x_{j-1})$ . Then let us repeat this procedure with the last  $k-1$  columns, and so on. Finally,

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \prod_{1 \leq j_1 < j_2 \leq m} (x_{j_2} - x_{j_1}) \times$$

$$\begin{vmatrix} 0 & \dots & Q[n-k] & \dots & Q[x_{n-k}, \dots, x_n] \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & Q[x_{k+2}, \dots, x_{n-k}] & \dots & Q[x_{k+2}, \dots, x_n] \\ Q[k+1] & \dots & Q[x_{k+1}, \dots, x_{n-k}] & \dots & Q[x_{k+1}, \dots, x_n] \\ Q[x_k, x_{k+1}] & \dots & Q[x_k, \dots, x_{n-k}] & \dots & Q[x_k, \dots, x_n] \\ \vdots & \dots & \vdots & \dots & \vdots \\ Q[x_1, \dots, x_{k+1}] & \dots & Q[x_1, \dots, x_{n-k}] & \dots & Q[x_1, \dots, x_n] \end{vmatrix}. \quad (21)$$

Since,  $\prod_{1 \leq j_1 < j_2 \leq n} (x_{j_2} - x_{j_1}) = (-1)^{n(n-1)/2} \prod_{1 \leq j_1 < j_2 \leq n} (x_{j_1} - x_{j_2})$ , one obtains (8) from (6), (15) and (21).

**The second case.** If  $n - k < k - 1$ , then, transforming the determinant similarly to the first case, from (17) one obtains:

$$D = (-1)^{\frac{(n+k+4)(n-k-1)}{2}} \det(C) \prod_{\substack{1 \leq j_1 \leq n-k-1 \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}),$$

where the columns of  $C$  are

$$C_i = \begin{pmatrix} Q[n-k-1+i] \\ Q[1, n-k-1+i] \\ \vdots \\ Q[1, \dots, n-k-1, n-k-1+i] \\ x_{n-k-1+i}^{2k-n} \\ \vdots \\ x_{n-k-1+i} \\ 1 \end{pmatrix}, \quad i = 1, \dots, k+1.$$

After subtracting from the first  $n - k$  rows the last row multiplied by the first element of the row, from the  $n - k + 1$ th to  $k$ th rows the next row, multiplied by  $x_{n-k}$ , and pulling out the common factors  $x_{n-k+1} - x_{n-k}, \dots, x_n - x_{n-k}$  one gets

$$D = (-1)^{\frac{(n+k+3)(n-k)}{2}} \det(C) \prod_{\substack{1 \leq j_1 \leq n-k \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}),$$

$$C_i = \begin{pmatrix} Q[n-k, n-k+i] \\ Q[n-k-1, n-k, n-k+i] \\ \vdots \\ Q[1, \dots, n-k, n-k+i] \\ x_{n-k+i}^{2k-n-1} \\ \vdots \\ x_{n-k+i} \\ 1 \end{pmatrix}, \quad i = 1, \dots, k.$$

Continuing this process, after  $2k - n$  steps one obtains:

$$D = (-1)^{\frac{(k-1)(2n-k+4)}{2}} \det(C) \prod_{\substack{1 \leq j_1 \leq k-1 \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1})$$

$$C_i = \begin{pmatrix} Q[n-k, \dots, k-1, k-1+i] \\ Q[n-k-1, \dots, k-1, k-1+i] \\ \vdots \\ Q[1, \dots, k-1, k-1+i] \\ 1 \end{pmatrix}, \quad i = 1, \dots, n-k+1.$$

After subtracting the first column from the next columns and pulling out the common factors  $x_{k+1} - x_k, \dots, x_n - x_k$  one gets

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \det(C) \prod_{\substack{1 \leq j_1 \leq k \\ j_1 < j_2 \leq n}} (x_{j_2} - x_{j_1}),$$

$$C_i = \begin{pmatrix} Q[n-k, \dots, k, k+i] \\ Q[n-k-1, n-k, \dots, k, k+i] \\ \vdots \\ Q[1, \dots, k, k+i] \end{pmatrix}, \quad i = 1, \dots, n-k.$$

Finally, one has to subtract the first column from the next columns and pull out the common factors  $x_{k+2} - x_{k+1}, \dots, x_n - x_{k+1}$ , then one has to subtract the second column from the next columns and pull out the common factors  $x_{k+3} - x_{k+2}, \dots, x_n - x_{k+2}$ , and so on. Then one gets

$$D = (-1)^{\frac{k(2n-k+3)}{2}} \prod_{1 \leq j_1 < j_2 \leq n} (x_{j_2} - x_{j_1})$$

$$\times \begin{vmatrix} \Delta_{n-k,k+1}(Q) & \Delta_{n-k,k+2}(Q) & \dots & \Delta_{n-k,n}(Q) \\ \Delta_{n-k-1,k+1}(Q) & \Delta_{n-k-1,k+2}(Q) & \dots & \Delta_{n-k-1,n}(Q) \\ \vdots & \vdots & \dots & \vdots \\ \Delta_{1,k+1}(Q) & \Delta_{1,k+2}(Q) & \dots & \Delta_{1,n}(Q) \end{vmatrix}. \quad (22)$$

Then, (15) and (22) imply (8).

Thus, Theorem 2 has proven in the case of simple and distinct roots of polynomials  $P$  and  $Q$ .

To prove the theorem in the general situation, we show first that it remains true if polynomials have common roots.

Let  $P(x) = p_n \prod_{j=1}^n (x - x_j)$  and  $Q(x) = q_m \prod_{j=1}^r (x - x_j) \prod_{j=1}^{m-r} (x - y_j)$  for some  $r$ ,  $0 < r \leq \min(m, n)$ . Then for sufficiently small  $\varepsilon$

polynomials  $P(x)$  and  $Q_\varepsilon(x) = q_m \prod_{j=1}^r (x - x_j - \varepsilon) \prod_{j=1}^{m-r} (x - y_j)$  have distinct roots and therefore (8) holds. Since both sides in (8) are continuous functions of  $\varepsilon$ , the formula remains true when  $\varepsilon \rightarrow 0$ .

Similarly one can prove (8) in the case of multiple roots. Denote by  $r$  the highest multiplicity of the roots of  $P(x)$ . We will use induction with respect to  $r$ . If  $r = 1$  the roots of  $P(x)$  are simple. Assume that (8) is true for some  $r$  and prove it for  $r + 1$ . Assume that there is one root of multiplicity  $r + 1$ . (The case of several roots of multiplicity  $r + 1$  can be proved similarly.) Let  $P(x) = (x - x_1)^{r+1} P_1(x)$ , where  $P_1(x)$  is a polynomial of degree  $n - r - 1$  with the roots distinct from  $x_1$ . Let  $P_\varepsilon(x) = (x - x_1 - \varepsilon)(x - x_1)^r P_1(x)$  for sufficiently small  $\varepsilon$ . By assumption, (8) is true for  $P_\varepsilon$  and  $Q$ . The left hand side in (8) is a continuous function of  $\varepsilon$ . In the right hand side

$$\lim_{\varepsilon \rightarrow 0} g[x_1 + \varepsilon, \underbrace{x_1, \dots, x_1}_r, x_2, \dots, x_s] = g[\underbrace{x_1, \dots, x_1}_{r+1}, x_2, \dots, x_s].$$

This observation completes the proof of Theorem 2 in the case of multiple roots.

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